Lecture 35.

Recall that a sequence of RU's $\{X_n\}$ with distribution functions $F_n(t)$ "converges in distribution" to a RU X with c.d.f. $F(t)$ iff

$$\lim_{n\to\infty} F_n(t) = F(t) \quad \forall t \text{ where } F(t) \text{ is continuous.}$$

Lemma: Suppose X_n has moment generating function $M_{X_n}(t)$. If $M_{X_n}(t) \to M_X(t)$ $\forall t$, then X_n converges in distribution to X.

Theorem (Central Limit Theorem)

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed RU's with mean μ and variance σ^2. Then

$$Z_n = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

converges in distribution to $Z = N(0,1)$.

Proof: We first consider the situation where the X_i are identically distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$.

- We assume that the moment generating function exists and is finite, $M(t) = E[e^{tX_i}]$ $\forall i$.

- Observe that the moment generating function of $X_i/\sqrt{n}$ is $E\left[e^{tX_i/\sqrt{n}}\right] = M\left(\frac{t}{\sqrt{n}}\right)$.

- By independence, the moment generating function of $Z_n = \sum_{i=1}^{n} \frac{X_n}{\sqrt{n}}$ is $\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$

- consider the function

$$L(t) = \log(M(t))$$

this is called the cumulant generating function. $L^{(n)}(0)$ is the n^{th} cumulant.

- Observe that:

$$L(0) = 0$$

$$L'(0) = \frac{m'(0)}{m(0)} = \frac{\mu}{1} = 0.$$

$$L''(0) = \frac{m(0)m''(0) - (m'(0))^2}{m(0)^2} = \frac{Var(X_i)}{1} = \sigma^2 = 1.$$

<u>Goal:</u> $\left[m(t/\sqrt{n})\right]^n \longrightarrow e^{t^2/2} = E\left[e^{tZ}\right]$ as $n \to \infty$.

<u>Equiv:</u> $nL(t/\sqrt{n}) \longrightarrow t^2/2$ as $n \to \infty$.

Now, we use some calculus:

$$\lim_{n\to\infty} n\, L(t/\sqrt{n}) = \lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}}$$

$$= \lim_{n\to\infty} \frac{L'(t/\sqrt{n})\, n^{-3/2}\, t\,(-1/2)}{-n^{-2}} \quad \text{(L'Hôpital)}$$

$$= \lim_{n\to\infty} \frac{L'(t/\sqrt{n})\, t}{2n^{-1/2}}$$

$$= \lim_{n\to\infty} \frac{L''(t/\sqrt{n})\,\cancel{(-1/2)}\,\cancel{n^{-3/2}}\, t^2}{2\,\cancel{(-1/2)}\,\cancel{n^{-3/2}}} \quad \text{(L'Hôpital)}$$

$$= \lim_{n\to\infty} L''(t/\sqrt{n})\,\frac{t^2}{2}$$

$$= L''(0)\,\frac{t^2}{2} = \frac{t^2}{2},$$

as required.

For the case where σ^2, μ are arbitrary, apply the above case to $X_i^* := \dfrac{(X_i - \mu)}{\sigma}$.

Review session begins.

Ex: Let X be the uniform RV on $(2,3)$. Let Y be uniform on $(4,6)$.

Assume X and Y are independent. Find the density function of $X+Y$.

Solution: Then density function of X and Y are

$$f_X(x) = \begin{cases} \frac{1}{3-2} = 1, & 2 < x < 3 \\ 0 & o/w. \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{6-4} = \frac{1}{2} & 4 < y < 6 \\ 0 & o/w. \end{cases}$$

The density function of $X+Y$ is the <u>convolution</u> of f_X and f_Y:

$$f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\,dx.$$

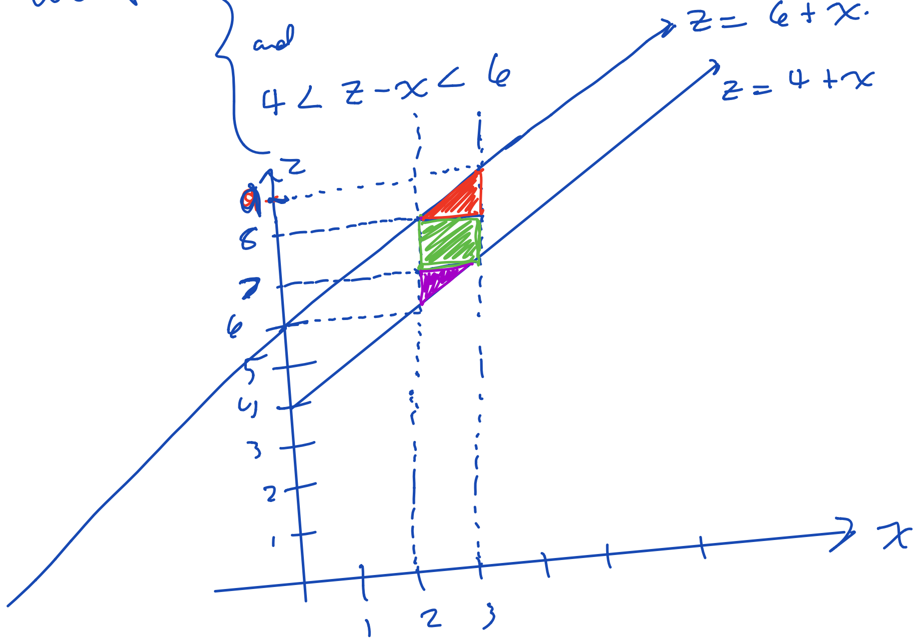
these end up the same.

$$= \int_2^3 f_Y(z-x)\, 1\, dx$$

$$f_Y(z-x) = \begin{cases} 1/2 & \text{iff } 4 < z-x < 6 \\ 0 & o/w. \end{cases}$$

so the integrand is non-zero iff $2 < x < 3$ and $4 < z-x < 6$.

Want: $\begin{cases} 2 < x < 3 \\ \text{and} \\ 4 < z - x < 6 \end{cases}$



Three regions:

when $6 < z \le 7$, we have $2 < x < z - 4$.

when $7 < z < 8$, we have $2 < x < 3$.

when $8 \le z < 9$ we have $z - 6 < x < 3$

So $$f_{X+Y}(z) = \begin{cases} \int_2^{z-4} \frac{1}{2}\, dx & 6 < z \le 7 \\ \int_2^{3} \frac{1}{2}\, dx & 7 < z < 8 \\ \int_{z-6}^{3} \frac{1}{2}\, dx & 8 \le z < 9 \end{cases}$$

$$= \begin{cases} \dfrac{z-6}{2} & 6 < z \leq 7 \\ \dfrac{1}{2} & 7 < z < 8 \\ \dfrac{9-z}{2} & 8 \leq z < 9 \\ 0 & \text{otherwise} \end{cases}$$